Precoder Design for Improving the Performance of MUSIC-based Angle-of-Arrival Estimator

Li Zhang*, Yong Huat Chew†, and Wai-Choong Wong*
*Department of Electrical and Computer Engineering - National University of Singapore
†Institute for Infocomm Research - Agency for Science, Technology and Research
Email: g0901835@nus.edu.sg, chewyh@i2r.a-star.edu.sg, wong_lawrence@nus.edu.sg

Abstract—In this paper, we propose a novel algorithm to improve the accuracy in estimating the angle-of-arrival (AoA) when the MUSIC (MUltiple SIgnal Classification) algorithm is used. An optimal precoder, with the objective to minimize the estimation errors of the AoAs, is first derived. However, to compute the optimal precoder requires the channel state information (CSI) exclusive of the receiver array, which cannot be separately estimated practically. A more feasible precoder design approach, which leverages on the feedback instantaneous CSI estimated at the receiver, is next proposed. Using the ideal optimal design to benchmark the performance, our simulation shows that the proposed practical precoder can achieve near-optimal performance, and both can achieve about 4-6 dB improvement in signal-to-noise ratio (SNR) compared to the case when only MUSIC algorithm is applied without precoding. Finally, the performance of the AoA estimator under channel estimation errors is studied through simulation, to demonstrate the robustness of the proposed approach.

I. INTRODUCTION

Accurate AoA estimation has received a significant amount of attention over the last few decades. It is a fundamental problem in many engineering applications, including wireless communications, radar, radio astronomy, sonar, navigation and tracking of objects. Various high resolution algorithms have been proposed, among which the subspace based algorithms are the most well known and are frequently used due to its low computational complexity and high resolution. The MUSIC [1] algorithm is a representative of subspace based algorithms. There has been considerable amount of interest in either analyzing the performance [2]–[4] or developing more advanced robust MUSIC algorithms [5], [6]. In the multipath environment, the MUSIC algorithm can be used to estimate the AoAs of signals impinging on the receive antenna array transmitting over multiple paths simultaneously [7]. However, all the efforts to improve the estimation accuracy in previous research works are spent on the processing of the received signal at the receiver. It is therefore natural to raise a question: is it possible to improve the estimation accuracy by pre-processing the transmitted signal based on the feedback channel side information at the transmitter (CSIT)? In this paper, we aim to find a strategy that exploits the CSIT to enhance the AoA estimation accuracy.

The multiple-input multiple-output (MIMO) technology has enabled a significant increase in the data transmission rate as well as improving the link reliability. By taking the benefit of CSIT, the performance of MIMO can be further enhanced. Precoder exploits the available CSIT and is a powerful processing technique applied on the signal before transmitted [8]. For stationary or slow time-varying channel, the channel information can be tracked instantaneously. On the other hand, for fast fading channel, CSIT is usually provided in terms of channel statistics, such as channel mean and covariance matrix which are also named mean CSIT and covariance CSIT respectively [8]. In the literature, the precoding design has been studied for various CSIT scenarios. In [9], perfect CSIT is used to compute the achievable channel capacity. The use of mean CSIT is studied in [10]–[12], while covariance CSIT is used in [10], [11], [13]. In [14], the authors proposed an algorithm in which both mean and transmit covariance CSIT are used.

In this paper, we propose a precoder design strategy with the objective to minimize the errors of AoA estimates derived in [2] when MUSIC algorithm is used. The system under consideration is a point-to-point wireless transmission system with stationary or slow-varying flat fading multipath propagation channel. We assume that the instantaneous CSI can be made available at the transmitter via feedback from the receiver. The ideal optimal precoder requires CSI knowledge without the contribution from the receiver array, which cannot be estimated practically. We further develop a more practical precoder design algorithm, which uses the CSI directly estimated at the receiver. The performance of the practical precoder is benchmarked against the optimal precoder design, and we demonstrate via simulation that near-optimal performance can be achieved. Further study looks into the improvement in performance over when only MUSIC algorithm is used without precoding. Finally, the robustness of the developed algorithm to channel estimation errors is also presented.

The rest of the paper is organized as follows. Section II introduces the system model. Both the principles of the practical and optimal precoder designs are derived in Section III. In Section IV, simulation results are presented and the accuracy of the two designs are compared with the estimation without precoding. Finally, Section V concludes the paper.

II. SYSTEM MODEL

The system under consideration is shown in Fig. 1. The channel is assumed to be stationary or slow-fading. The relative delays between any two multipaths are assumed to be much less than the bit duration, hence, flat fading channel is
assumed. The precoder is first removed (i.e., set to an identity matrix) and the receiver performs channel estimate and the instantaneous CSI is feedback to the transmitter through a noise-free feedback channel. The precoder matrix $F$ is then computed according to the CSI, and the theory behind the design will be presented in Section III.

The total transmission power is $P_w$. The input signal is $x(t)$ which consists of $L$ signals each comprising ideal rectangular pulses modulated by a pseudo-noise (PN) sequence. The number of input sequences must be larger than or equal to the total number of paths $L$ so that all the paths are distinguishable at the receiver, however, for simplicity, we choose the smallest number $L$ in this paper. Generally, $x(t)$ can be any correlated signals with known covariance. Then a de-correlation matrix, which is easy to obtain, is needed before the precoder, so that the input signals to the precoder are uncorrelated and allocated with equal power. In this paper, the de-correlation matrix is not considered, so $x(t)$ is assumed to be uncorrelated signals allocated with equal power $P_w/L$ given by

$$x_i(t) = \sum_{k=-\infty}^{\infty} \sqrt{\frac{P_w}{L} a_{i,k} \text{rect}_T(t-kT)}, i = 1, \ldots, L$$

where $T$ denotes the bit duration, $a_{i,k} \in \{-1, +1\}$, and $\text{rect}_T(t)$ is a rectangular pulse of unit amplitude and duration $T$. The output of the precoder is $s(t)$, which is expressed as $s(t) = Fx(t)$.

For simplicity, we assume that both the transmitter and receiver are each equipped with a uniform linear array (ULA). However, the proposed strategy is applicable for arbitrary array shape. The ULAs at the transmitter and the receiver include $M$ and $N$ antennas, respectively, spaced apart by $d = \lambda_s/2$, where $\lambda_s$ is the wavelength. The total number of paths is $L$, with each path containing the AoD $\Omega_{T,i}$, AoA $\Omega_{R,i}$, and complex gain $\alpha_i$. As assumed in [1], $L$ should not exceed $\min(M, N)$. Since this is a flat-fading channel, the time delay is relatively smaller than the bit duration and hence will be neglected in the expression.

For a ULA, the array manifold is expressed as $C(\Omega) = [C_1(\Omega), \ldots, C_n(\Omega)]^T$, where $n$ is the antenna number, $\Omega$ is the AoA or AoD, $(\cdot)^T$ is the transpose of the argument, and $C_i(\Omega) = \exp[j2\pi(i-1)d\cos\Omega/\lambda_s], i = 1, \ldots, n$. Thus, for the $l$th propagation path, the $j$th transmit antenna gain response due to the AoD $\Omega_{T,j}$ is $C_{T,j}(\Omega_{T,j})$ and the $i$th receive antenna gain response due to the AoA $\Omega_{R,i}$ is $C_{R,i}(\Omega_{R,i})$. So the received signal for the $i$th antenna is

$$r_i(t) = \sum_{j=1}^{M} \sum_{l=1}^{L} \alpha_l C_{T,j}(\Omega_{T,j}) C_{R,i}(\Omega_{R,i}) s_j(t) + n_i(t)$$

where $n_i(t)$ is the complex white Gaussian noise with mean zero and variance $\sigma_n^2$.

The channel model may be expressed in matrix form as

$$r(t) = Hs(t) + n(t)$$

where

$$H = [C_R(\Omega_{R,1}), \ldots, C_R(\Omega_{R,L})]^T$$

$$= [C_T(\Omega_{T,1}), \ldots, C_T(\Omega_{T,L})]^T$$

We define $H_0 = H_0 C_T(\Omega_T)$, which will be used in the derivation of optimal precoder. The signal impinging on the receive antenna array is $y(t)$, and is expressed as $y(t) = H_0 s(t)$. For convenience, hereafter the time index $t$ will be dropped.

The receiver is assumed to be continuously estimating the channel, so that the instantaneous CSI is known at the receiver. The channel estimation may be expressed as $H = H + \hat{E}$, where $H$ is the estimate of $H$, and $\hat{E}$ is the estimation error matrix comprising independent identically distributed (i.i.d.) complex Gaussian variables with zero mean and variance $\sigma_E^2$. As in [15], $\Psi^2 = ||H||^2_F/(MN)$, where $||.||_F$ is the Frobenius norm, is defined as the channel normalization factor. $\sigma_E^2$ is the quality of the channel estimation, for instance, $\sigma_E^2 = 0.1$ indicates 10% estimation error. The MUSIC algorithm is used to estimate the AoAs of the $L$ paths.

### III. PRECODER DESIGN

In this section, the precoder design strategy with the objective to minimize the estimation error of the MUSIC algorithm derived in [2] is presented. As the optimal solution cannot be obtained practically, an approximate approach, which is later shown through simulation to be able to achieve near-optimal performance, is next proposed.
A. Theory

As shown in [2], when $N$ increases, the sum of variances of $L$ AoAs of the MUSIC estimation tends to the following limit

$$\text{var}_{MU}(\Omega_R) = \frac{6\sigma_n^2}{IN} \text{tr}(P^{-1})$$

(4)

where $\Omega_R$ consists of the AoAs of $L$ paths, $I$ is the number of samples, $P = E[yy^H]$ is the covariance matrix of $y$, and $\text{tr}(\cdot)$ and $(\cdot)^H$ are the matrix trace and Hermitian transpose of the argument, respectively.

Thus, with the aim to minimize the error variance, we formulate the design problem under transmit power constraint as follow:

$$\min_F \quad \text{tr}(P^{-1}) = \frac{L}{P_w} \text{tr}\left(\left(H_hFF^H H_h^H\right)^{-1}\right)$$

subject to $\frac{P_w}{L} \text{tr}(FF^H) \leq P_w$\hspace{1cm} (5)

B. Practical Approach

The solution to (5) gives the optimal precoding matrix to obtain minimum error variance defined in (4). However, the CSI $H_h$, which is required at the transmitter may not be able to obtain practically, rather, most of the channel estimation performed at the receiver estimate the channel matrix $H$. Thus, we propose a more practical design given by

$$\min_F \quad \frac{L}{P_w} \text{tr}\left(\left(H_hFF^H H_h^H\right)^{-1}\right)$$

subject to $\frac{P_w}{L} \text{tr}(FF^H) \leq P_w$\hspace{1cm} (6)

where $(\cdot)^\dagger$ is the Moore-Penrose pseudoinverse of the argument. The reason that the inverse is replaced by the pseudoinverse is that the argument is not full rank.

The solution to (6) can be shown later through simulation that the performance is closed to that for optimal precoder design.

C. Solution

The approaches to solve (5) and (6) are similar, except either $H_h$ or $H$ is used. In this subsection, we demonstrate using the practical design.

**Theorem 1.** Let the truncated singular value decomposition (SVD) of $H$ be given by $H = U_h \Lambda_h V_h^H$, where $U_h$ and $V_h$ are $N \times L$ and $M \times L$ matrices with the property $U_h^H U_h = I$ and $V_h^H V_h = I$, respectively, and $\Lambda_h$ is a $L \times L$ diagonal matrix with the singular values of $H$ permuted in decreasing order as the diagonal elements. Then the precoder that minimizing (6) may be expressed as $F = V_h D$, where $D$ is a diagonal matrix.

**Proof:** For convenience, we omit the constant in (6). Thus our objective is to minimize a matrix trace subject to the power constraint. Substituting the truncated SVD into the matrix trace in (6), we have

$$\text{tr}\left(\left(H hFF^H H h^H\right)^\dagger\right) = \text{tr}\left(\left(U_h \Lambda_h V_h^H FF^H V_h \Lambda_h^H U_h^H\right)^\dagger\right)$$

(7)

Since $U_h$ is full column rank, and $\left(A_h V_h^H FF^H V_h \Lambda_h^H\right)$ is full rank, (7) can be simplified as

$$\text{tr}\left(\left(U_h \Lambda_h V_h^H FF^H V_h \Lambda_h^H U_h^H\right)^\dagger\right) = \text{tr}\left(\left(U_h \Lambda_h V_h^H FF^H V_h \Lambda_h^H\right)^\dagger U_h^\dagger\right)$$

$$= \text{tr}\left(\left(A_h V_h^H FF^H V_h \Lambda_h^H\right)^\dagger\right)$$

(8)

For full rank square matrix, the pseudoinverse and inverse are identical, thus

$$\text{tr}\left(\left(A_h V_h^H FF^H V_h \Lambda_h^H\right)^\dagger\right) = \text{tr}\left(\left(A_h V_h^H FF^H V_h \Lambda_h^H\right)^{-1}\right)$$

$$= \text{tr}(A_h^{-2})$$

(9)

where $\Sigma = V_h^H FF^H V_h$.

Next, we need to prove that $\Sigma$ is diagonal. We use the following lemma [16].

**Lemma 1.** If $U$ and $V$ are $n \times n$ positive semidefinite Hermitian matrices with eigenvalues $\lambda_i(U)$ and $\lambda_i(V)$, respectively, arranged in decreasing order, and $\lambda_i(UV)$ are the eigenvalues of the product matrix $UV$, then

$$\text{tr}(UV) = \sum_{i=1}^{n} \lambda_i(UV) \geq \sum_{i=1}^{n} \lambda_i(U) \lambda_{n-i+1}(V)$$

(10)

From the proof in [13], we know that if matrix $U$ in Lemma 1 is diagonal, the equality in (10) holds only when matrix $V$ is also diagonal, and vice versa. In addition, the arrangement of diagonal elements of the two matrices should be opposite. Applying Lemma 1 and the above conclusion to (9), the fact that $\Lambda_h^{-2}$ is diagonal implies that $\Sigma^{-1}$ must be diagonal, so that the equality in (10) holds and (9) obtains its minimum. So $\Sigma$ is also diagonal. Besides, the diagonal elements of $\Sigma^{-1}$ must be arranged in the inverse order of those of $\Lambda_h^{-2}$. Thus, we have

$$V_h^H FF^H V_h = \Sigma$$

$$\Rightarrow FF^H = V_h \Sigma V_h^H$$

$$\Rightarrow F = V_h \Sigma^{1/2} = V_h D$$

(11)

Thus, Theorem 1 is proved.

According to the form of $F$, the precoder is decomposed into two parts, namely the power allocation matrix $D$ and the beamforming matrix $V_h$. The structure is depicted in Fig. 2.

Next, we will derive the expression for the power allocation matrix $D$ that minimizes (6).

Let the two diagonal matrices $D = \text{diag}\{d_1, \ldots, d_L\}$ and $\Lambda_h = \text{diag}\{\lambda_{h,1}, \ldots, \lambda_{h,L}\}$, where $\lambda_{h,i}, i = 1, \ldots, L$ are the singular values of $H$, then combining (7) and (9), we have

$$\text{tr}\left(\left(H hFF^H H h^H\right)^\dagger\right) = \text{tr}(A_h^{-2}D^{-2}) = \sum_{i=1}^{L} \frac{1}{\lambda_{h,i}^2 d_i^2}$$

(12)
The power constraint can be simplified to \( \text{tr}(\mathbf{FF}^H) = \sum_{i=1}^L d_i^2 \leq L \). The simplified objective function may then be expressed as

\[
\min_{d_1, \ldots, d_L} \sum_{i=1}^L \frac{1}{\lambda_{h,i}} d_i^2 \quad \text{subject to } \sum_{i=1}^L d_i^2 \leq L \quad (13)
\]

In order to solve (13), we apply the Lagrange multiplier method. By multiplying a nonnegative number to the constraint, the cost function is written as

\[
L(\lambda, d_1, \ldots, d_L) = \sum_{i=1}^L \frac{1}{\lambda_{h,i}} d_i^2 + \lambda \left( \sum_{i=1}^L d_i^2 - L \right) \quad (14)
\]

where \( \lambda \geq 0 \) is the Lagrange multiplier associated with the power constraint.

The cost function in (14) is minimized when the derivatives of \( L(\lambda, d_1, \ldots, d_L) \) over \( d_i, i = 1, \ldots, L \), are all equal to zero. Thus we have

\[
\frac{\partial L}{\partial d_i} = -\frac{2}{\lambda_{h,i}^2} d_i + 2 \lambda d_i = 0, \quad i = 1, \ldots, L \quad (15)
\]

Solving (15) and determining the constant \( \lambda \) from the equality \( \sum_{i=1}^L d_i^2 = L \), we obtain the optimum power allocation as

\[
d_i = \sqrt{\frac{L}{\lambda_{h,i} \sum_{j=1}^L 1/\lambda_{h,j}}}, \quad i = 1, \ldots, L \quad (16)
\]

The last step is to verify that the arranging order of diagonal elements of \( \Sigma^{-1} \) is identical to the inverse of the permutation of the diagonal elements of \( \Lambda^{-2}_h \), so that the equality in (10) holds. Based on the fact that \( \Sigma = \mathbf{D}^2 \) and the expression of (16), it is easy to see that the diagonal elements of \( \Sigma^{-1} \) are arranged in decreasing order, while those of \( \Lambda^{-2}_h \) are in increasing order. Thus, the whole derivation is complete.

We conclude that the practical precoder is given by

\[
\mathbf{F}_{\text{prec}} = \mathbf{V}_h \mathbf{D} \quad (17)
\]

where \( \mathbf{D} \) is diagonal and the diagonal elements are given by (16). In practice, with instantaneous CSIT, the precoder \( \mathbf{F} \) is designed according (17).

Using the similar derivations, we are able to solve (5), which gives the optimal solution as a benchmark of performance. The truncated SVD of \( \mathbf{H}_b \) is given by \( \mathbf{H}_b = \mathbf{U}_b \mathbf{A}_b \mathbf{V}_b^H \), where \( \mathbf{A}_b = \text{diag}\{\lambda_{b,1}, \ldots, \lambda_{b,L}\} \) are the singular values of \( \mathbf{H}_b \) arranged in decreasing order, then the solution of (5) is given by

\[
\left\{ \begin{array}{l}
\mathbf{F}_{\text{opt}} = \mathbf{V}_b \mathbf{D}_b \\
\mathbf{d}_{b,i} = \sqrt{\frac{L}{\lambda_{b,i} \sum_{j=1}^L 1/\lambda_{b,j}}}, \quad i = 1, \ldots, L
\end{array} \right. \quad (18)
\]

where \( \mathbf{D}_b = \text{diag}\{d_{b,1}, \ldots, d_{b,L}\} \).

Since the optimal precoder requires information of \( \mathbf{H}_b \) which is not obtainable in the system considered in this paper, (18) is only used in the simulation to benchmark the performance.

IV. SIMULATION AND PERFORMANCE ANALYSIS

The performance of the proposed algorithm is evaluated in synthetically generated propagation channel. The transmitter and receiver are both using ULAs with six elements separated by half wavelength. The channel is assumed to have three propagation paths. The AoAs, AoDs, and the amplitude of complex path gains are listed in Table I. The input signal of the precoder consists of three independent PN sequences each with 128 rectangular pulses. The number of samples taken at the receiver is 32. The discretization steps of the MUSIC algorithm is 0.02° for AoA estimation.

The simulation consists of two parts. In the first part, three strategies are applied, namely no precoding, practical precoding and optimal precoding, respectively. The variance of channel estimation error \( \sigma^2 \) is set to 0, i.e. the channel estimation at the receiver is error-free. For the strategy without precoding, the three PN sequences are sent to the first three antennas and the last three antennas, respectively. In the second part, the performances of practical precoding strategy with different channel estimation errors are simulated.

In the simulation, we compute the root mean square error (RMSE) of the estimated AoAs with the SNR changing from -10 to 20 dB. Since we fix the transmission power at the transmitter, the SNR referred to here is the ratio between the transmission power and the noise power at the receiver.

From Fig. 3, we can see that the precoding strategy, either the practical or optimal, achieves 4dB improvement on the accuracy, compared with the estimation without precoding, when the SNR is high. At low SNR, the performance of the precoding strategy is more than 6dB better than that without precoding. The simulation also reveals that the practical strategy achieves the performance which is nearly identical to the optimal solution. This indicates the efficiency of our design using the conventional channel estimation techniques.

The performance of the practical strategy with different channel estimation errors is presented in Fig. 4. We can see...
that at high SNR the performance of AoA estimation is slightly degraded with the increase of the variance of the channel estimation error $\sigma_e^2$. At low SNR, the improvement is less predictable, however, even with 50% channel estimation error, i.e. $\sigma_e^2=0.5$, improvement can still be observed. This indicates the robustness of the precoder especially when variance of the channel estimate error is less than 20%.

V. CONCLUSION AND FUTURE WORK

In this paper, we propose a practical precoder design strategy with the objective of minimizing the AoA estimation error of the MUSIC algorithm. We also derive the optimal precoder with similar derivations which benchmarks the performance. Simulation result shows that the precoder results in 4-6 dB improvement in AoA estimation error as compared with the estimation without precoding. In addition, the near identical performance between the practical and the optimal designs indicates the practicability of our approach since any channel estimation technique can be used. In the scenarios with channel estimation error, the performance is only slight degraded, especially if the variance of channel estimation error is less than 20%.

In the future, the precoder design in other scenarios will be studied. When the channel is fast fading, only the statistical CSIT, i.e. channel mean and/or covariance, is available. It is challenging to design the precoder under such conditions. Another aspect is the precoder design in frequency-selective channel either with single carrier or multiple carriers. The third aspect is to develop precoders for other AoA estimation algorithms. Corresponding objective functions are to be proposed according the expressions of estimation error variance [4].

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